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LOCALIZED ASYMPTOTIC SOLUTIONS OF THE NAVIER–STOKES EQUATIONS AND LAMINAR WAKES IN AN INCOMPRESSIBLE FLUID†

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Equations are derived to describe the far-field laminar wake behind a body in incompressible fluid flow with an arbitrary distribution of the free-stream (unperturbed flow) velocity. For certain classes of free-stream flows, analysis of these equations enables various processes in narrow wakes or jets to be described (the interaction of the longitudinal transverse velocity components in a jet, cause it to accelerate or decelerate and conservation of the energy of the wake by distortion of its trajectory regardless of viscous dissipation). In particular, conditions are obtained for the wake growth in spiral flows, analogous to the Rayleigh conditions for the instability of two-dimensionally radially symmetric flows relative to three-dimensional short-wave perturbations.

Mathematically speaking, the hydrodynamic problems under consideration are characterized by the presence of a small parameter ε—the ratio of the characteristic transverse scale to the longitudinal scale The equations derived and investigated below determine the principal term in the asymptotic expansion of the solution of the Navier–Stokes equations in terms of this small parameter. It is assumed that the ratio of the traverse components of the velocity to the longitudinal component is also small (of the order of ε), as is the viscosity coefficient (of the order of ε^2—if the viscosity is appreciable, it instantaneously destroys the wake).

A scheme for constructing such asymptotic expansions was proposed in [1], based on an analysis of the resonance properties of the Navier–Stokes equations (cf. [1–5]). This scheme has been implemented for the equations of magnetohydrodynamics [6–9], producing equations that describe the asymptotic behaviour of localized fields in a plasma; these equations have already been derived for the case of a cylindrically symmetric pinch [10]. Some of the phenomena studied below in laminar wakes have analogies in plasma physics; for example, the interaction of the longitudinal and transverse velocity components in the wake is analogous to the Shafranov–Pustovitov effect (the interaction of the longitudinal and transverse components of the magnetic field in a plasma filament, see [9, 11]).

1. DERIVATION OF THE EQUATIONS DESCRIBING A LAMINAR WAKE

Consider the steady flow of an incompressible fluid. The velocity field $\mathbf{v}(x)$ satisfies the stationary Navier–Stokes equations

$$(\mathbf{v}, \nabla)\mathbf{v} + \nabla p = \frac{1}{\mathrm{Re}}\Delta\mathbf{v}, \quad (\nabla, \mathbf{v}) = 0 \tag{1.1}$$

where $p(\mathbf{x})$ is the pressure in the fluid and Re is the Reynolds number. Given a free-stream flow $\mathbf{V}(\mathbf{x})$, we wish to study perturbations of the flow with the following properties:

1. the perturbations are small, that is, the velocity field $\mathbf{v}$ differs only slightly from the free-stream velocity field $\mathbf{V}$;

2. the perturbations are localized in a small neighbourhood of a certain curve γ in three-dimensional space.

We introduce a small parameter ε characterizing the ratio of the width of the aforementioned neighbourhood to the characteristic scale of variation of the free-stream velocity field $\mathbf{V}(\mathbf{x})$. Conditions 1 and 2 mean that we are studying solutions of Eqs (1.1) that possess the following properties

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$$\lim_{\varepsilon\to 0}(\mathbf{v}-\mathbf{V})=0, \quad \lim_{\varepsilon\to 0}\frac{\mathbf{v}-\mathbf{V}}{\varepsilon}=0$$

for all $\mathbf{x}$ at a finite distance $\rho > 0$ (independent of ε) from γ.

Such solutions satisfy specially chosen boundary conditions for Eqs (1.1). Let us consider an arbitrary surface Γ transverse to the field $\mathbf{V}$. Suppose it is defined in R^3_x by the equations $\mathbf{x} = \mathbf{r}(\alpha)$, where $\alpha = (\alpha_1, \alpha_2)$ are coordinates on Γ and $\mathbf{r}(\alpha)$ are smooth functions such that the vectors $\partial\mathbf{r}/\partial\alpha_1$, $\partial\mathbf{r}/\partial\alpha_2$, $\mathbf{V}$ are linearly independent at all points of Γ. Choosing smooth functions $S^0_j(\alpha)$ ($j = 1, 2$) on Γ, we assume given, as boundary conditions on the surface, a small perturbation of the free-stream velocity field localized near a point α_0 at which these functions vanish

$$\mathbf{v}|_\Gamma = \mathbf{V}+\varepsilon\mathbf{U}_0\left(\frac{S^0_1(\alpha)}{\varepsilon},\frac{S^0_2(\alpha)}{\varepsilon},\alpha\right) \tag{1.2}$$

where $\mathbf{U}_0(y_1, y_2\ \alpha)$ is a smooth vector-valued function that decreases rapidly as $|\mathbf{y}| \to \infty$ ($y_j = S^0_j/\varepsilon$).

This specification of conditions (1.2) has the following meaning. It is assumed that a narrow wake is formed in the fluid, the velocity field of the perturbation being known in a certain area element transverse to the free stream. The problem is to describe the changes in the wake due to the flow as the distance upstream from the area element increases. Thus, we shall seek a solution of problem (1.1), (1.2) in the half-space bounded by the surface Γ.

We will seek the solution of problem (1.1), (1.2) in the form

$$\mathbf{v}(\mathbf{x},\varepsilon)=\mathbf{V}^0(\mathbf{x},\varepsilon)+\varepsilon\mathbf{U}\left(\frac{S_1(\mathbf{x})}{\varepsilon},\frac{S_2(\mathbf{x})}{\varepsilon},\mathbf{x}\right)+\ldots$$

$$p(\mathbf{x},\varepsilon)=p_0+\varepsilon\pi_0\left(\frac{S_1(\mathbf{x})}{\varepsilon},\frac{S_2(\mathbf{x})}{\varepsilon},\mathbf{x}\right)+\varepsilon^2\pi\left(\frac{S_1(\mathbf{x})}{\varepsilon},\frac{S_2(\mathbf{x})}{\varepsilon},\mathbf{x}\right)+\ldots \tag{1.3}$$

where $\mathbf{V}^0$ is a smooth vector field (the non-decreasing part of the solution), p_0 is the pressure corresponding to the field $\mathbf{V}^0$, the functions $\mathbf{U}(y_1, y_2, \mathbf{x})$, $\pi(y_1, y_2, \mathbf{x})$, $\pi_0(y_1, y_2, \mathbf{x})$ decrease $|\mathbf{y}| \to \infty$ more quickly than $|\mathbf{y}|^{-1}$, and the functions $S_j(\mathbf{x})$ vanish on a curve γ (which is not known in advance); the vectors ∇S_1 and ∇S_2 are linearly independent. Throughout, $\mathbf{y}$ is the two-dimensional vector of "stretched" variables $y_j = S_j/\varepsilon$. As already remarked (see above), the wake can only exist if the viscosity is sufficiently small; we therefore assume that $\mathrm{Re}^{-1} = \varepsilon^2\nu$, $\nu = O(1)$ in (1.1).

Substituting (1.3) into (1.1) and considering the resulting equation outside a neighbourhood of γ independent of ε, we obtain $\mathbf{V}^0 = \mathbf{V} + o(\varepsilon)$ (since we will be investigating asymptotic expansions with the same accuracy, we will assume throughout that $\mathbf{V}^0 = \mathbf{V}$). We now equate the coefficients of each power of ε to zero in the remaining terms.

For ε^0 we obtain

$$\sum_{j=1}^{2}(\mathbf{V},\nabla S_j)\frac{\partial\mathbf{U}}{\partial y_j}+\nabla S_j\frac{\partial\pi_0}{\partial y_j}=0, \quad \sum_{j=1}^{2}\frac{\partial}{\partial y_j}(\mathbf{U},\nabla S_j)=0 \tag{1.4}$$

Multiplying the first (vector) equation in (1.4) by ∇S_1 and ∇S_2, we have

$$\sum_{j=1}^{2}(\mathbf{V},\nabla S_j)\frac{\partial u_k}{\partial y_j}+(\nabla S_k,\nabla S_j)\frac{\partial\pi_0}{\partial y_j}=0, \quad k=1,2; \quad \sum_{j=1}^{2}\frac{\partial u_j}{\partial y_j}=0 \tag{1.5}$$

where $u_j = (\mathbf{U}, \nabla S_j)$. Differentiating the first equation in (1.5) ($k = 1$) with respect to y_1 and the second with respect to y_2, adding them and taking the third equation into consideration, we get

$$D^2\pi_0 \equiv \sum_{j,k=1}^{2}(\nabla S_j,\nabla S_k)\frac{\partial^2}{\partial y_j\partial y_k}\pi_0=0$$

Since this equation is elliptic and π_0 decreases at infinity, it follows that $\pi_0 = 0$ and we deduce from (1.5) that

$$(\mathbf{V}, \nabla S_j) = 0 \tag{1.6}$$

It follows from (1.6), in particular, that the curve γ (on which $S_1 = S_2 = 0$) is a streamline of the field **V**.

We will now consider the equations obtained by equating the coefficient of ε^1 to zero. Using (1.6) and the fact that π_0 is zero, we deduce from the first (vector) equation of (1.1) that

$$(\mathbf{V},\nabla)\mathbf{U}+(\mathbf{U},\nabla)\mathbf{V}+\sum_{j=1}^{2}\left((\mathbf{U},\nabla S_j)\frac{\partial \mathbf{U}}{\partial y_j}+\nabla S_j\frac{\partial \pi}{\partial y_j}\right)=\nu D^2\mathbf{U} \tag{1.7}$$

where ∇ is the gradient with respect to the "slow" variables **x** with **y** held constant. Putting $w = (\mathbf{U}, \mathbf{V})$, using (1.5) and the identities

$$\begin{aligned}
&(\nabla S_j,(\mathbf{V},\nabla)\mathbf{U})=(\mathbf{V},\nabla)u_j-(\mathbf{U},(\mathbf{V},\nabla)\nabla S_j)\\
&(\mathbf{V},\nabla)\nabla S_j=\nabla(\mathbf{V},\nabla S_j)-\frac{\partial V^*}{\partial x}\nabla S_j=-\frac{\partial V^*}{\partial x}\nabla S_j\\
&(\mathbf{V},(\mathbf{V},\nabla)\mathbf{U})=(\mathbf{V},\nabla)w-(\mathbf{U},(\mathbf{V},\nabla)\mathbf{V})=(\mathbf{V},\nabla)w-\left(\mathbf{U},\frac{\partial V}{\partial x}\mathbf{V}\right)\\
&\mathbf{U}=\sum_{i,j=1}^{2}(T^{-1})_{ij}u_i\nabla S_j+\frac{w\mathbf{V}}{|V|^2}
\end{aligned} \tag{1.8}$$

where $\partial V/\partial x$ is the 3×3 matrix with elements $\partial V_i/\partial x_j$ ($\partial V^*/\partial x$ denotes the transpose), and T is the 2×2 matrix with elements (∇S_i and ∇S_j), we deduce from (1.7) and (1.5) equations for the perturbation $\mathbf{u} = (u_1, u_2)$, w

$$\begin{aligned}
&\dot{\mathbf{u}}+(\mathbf{u},\nabla_y)\mathbf{u}=-T\nabla_y\pi+\nu D^2\mathbf{u}-AT^{-1}\mathbf{u}-\mathbf{a}w,\quad (\nabla_y,\mathbf{u})=0\\
&\dot{w}+(\mathbf{u},\nabla_y)w=\nu D^2 w-(T^{-1}\mathbf{b},\mathbf{u})
\end{aligned} \tag{1.9}$$

where ∇_y is the gradient with respect to **y**, $\dot{\mathbf{u}} = (\mathbf{V}, \nabla)\mathbf{u}$($\dot{w} = (\mathbf{V}, \nabla)w$), and the elements of the 2×2 matrix A and the vectors **a** and **b** are defined by

$$A_{ij}=2\left(\nabla S_i,\frac{\partial V}{\partial x}\nabla S_j\right)$$

$$a_j=\frac{2}{\mathbf{V}^2}\left(\nabla S_j,\frac{\partial V}{\partial x}\mathbf{V}\right),\quad b_j=\left(\mathbf{V},\left(\frac{\partial V}{\partial x}-\frac{\partial V^*}{\partial x}\right)\nabla S_j\right)$$

We have thus proved the following theorem.

Theorem. Let the vector field **V**, the curve γ and the functions S_j satisfy the following conditions:

1. $\mathbf{V}(\mathbf{x}, \varepsilon)$ is a smooth vector-valued function of $(\mathbf{x}, \varepsilon)$, which, as $|\mathbf{x}| \to \infty$, tends to a vector $\mathbf{V}_0(\varepsilon)$ in such a way that all the derivatives of **V** with respect to x_j tend to zero, and which satisfies Eqs (1.1) in $R_3 \bmod o(\varepsilon)$.
2. γ is a smooth non-self-intersecting trajectory of **V** which departs to infinity, and $|\mathbf{V}|_\gamma \geqslant \delta > 0$.
3. $S_j(\mathbf{x})$ are smooth functions, ∇S_1 and ∇S_2 are linearly independent in R_3, $\nabla S_j \to k_j = \text{const}$ as $|\mathbf{x}| \to \infty$, all the higher-order derivatives of S_j tend to zero $S_j|_\gamma = 0$ and Eqs (1.6) hold in some neighbourhood Ω of γ, independent of ε.

Let $\mathbf{u}(\mathbf{y}, \mathbf{x})$, $w(\mathbf{y}, \mathbf{x})$, $\pi(\mathbf{y}, \mathbf{x})$ be a smooth solution of system (1.9) for $\mathbf{y} \in \mathbf{R}^2$, $\mathbf{x} \in \Omega$, satisfying the conditions $\lim_{|\mathbf{y}|\to\infty}|\mathbf{u}| = \lim_{|\mathbf{y}|\to\infty}|w|$, extended continuously to $R^2_{\mathbf{y}} \times R^3_{\mathbf{x}}$ in such a way that $w \equiv 0$ in a neighbourhood of the singular points of the field $\mathbf{V}(\mathbf{x})$; let $\tilde{\mathbf{u}}(\mathbf{y}, \mathbf{x})$ be a smooth two-dimensional vector-valued function that decreases as $|\mathbf{y}| \to \infty$ and satisfies the following equality in Ω

$$(\nabla_y,\tilde{\mathbf{u}})=-(\nabla,\mathbf{U}) \tag{1.10}$$

(the function $\mathbf{U}(\mathbf{y}, \mathbf{x})$ is defined by (1.8)). Then the vector-valued function

$$\mathbf{V}(\mathbf{x},\varepsilon)+\varepsilon\mathbf{U}\left(\frac{S_1(\mathbf{x})}{\varepsilon},\frac{S_2(\mathbf{x})}{\varepsilon},\mathbf{x}\right)+\varepsilon^2\tilde{\mathbf{u}}\left(\frac{S_1(\mathbf{x})}{\varepsilon},\frac{S_2(\mathbf{x})}{\varepsilon},\mathbf{x}\right)$$

satisfies Eqs (1.1) in R^3 mod $o(\varepsilon)$.

Remarks. 1. It follows from the theorem that the construction of an asymptotic solution of problem (1.1), (1.2) reduces to solving a Cauchy problem for system (1.9). Indeed, let γ be a trajectory of $\mathbf{V}$ emanating from a point α_0 on Γ. A solution of Eqs (1.6) that satisfies the condition $S_j|_\Gamma = S_j^0$ has the form $S_j(\mathbf{x}) = S_j^0(\alpha(\mathbf{x}))$, where $\alpha(\mathbf{x})$ is the point at which the streamline of $\mathbf{V}$ passing through $\mathbf{x}$ intersects the surface Γ. In other words, if $\mathbf{X}(\alpha, t)$ is a solution of the system of ordinary differential equations

$$d\mathbf{X}/dt = \mathbf{V}(\mathbf{X}), \quad \mathbf{X}(0) = \mathbf{r}(\alpha)$$

then $\alpha(\mathbf{x})$ is found from the equations

$$X(\alpha, t) = \mathbf{x} \tag{1.11}$$

Now let $\mathbf{u}(\mathbf{y}, \alpha, t)$, $w(\mathbf{y}, \alpha, t)$ be the solution of the Cauchy problem for system (1.9) with initial data

$$\mathbf{u}|_{t=0} = \mathbf{u}^0(\mathbf{y}, \alpha), \quad w|_{t=0} = w^0(\mathbf{y}, \alpha) \tag{1.12}$$

Then, obviously, the vector $\mathbf{U}$ constructed from the functions $\mathbf{u}(\mathbf{y}, \alpha(\mathbf{x}), t(\mathbf{x}))$, $w(\mathbf{y}, \alpha(\mathbf{x}), t(\mathbf{x}))$ (see (1.8)) defines a perturbations of the field $\mathbf{V}$ that satisfies conditions (1.2). Here $\alpha(\mathbf{x})$ and $t(\mathbf{x})$ denotes a solution of system (1.11) ($t(\mathbf{x})$ is the time in which the trajectory of $\mathbf{V}$ emanating from the point $\alpha(\mathbf{x})$ on the surface Γ reaches $\mathbf{x}$).

2. The first equations of system (1.9) recall the two-dimensional stationary Navier–Stokes equations, with additional "forces" which depend linearly on the velocities; the last equation is similar to the heat conduction equation.

3. We say that a function $\tilde{\mathbf{u}}(\mathbf{x}, \varepsilon)$ satisfies system (1.1) mod $o(\varepsilon^2)$ if a function $p(\mathbf{x}, \varepsilon)$ exists such that $|(\tilde{\mathbf{u}}, \nabla)\tilde{\mathbf{u}} + \nabla\tilde{p} - \mathrm{Re}^{-1}\Delta\tilde{\mathbf{u}}| = o(\varepsilon^k)\ |(\nabla, \tilde{\mathbf{u}})| = o(\varepsilon^k)$.

4. Since $|\mathbf{U}| = o(1)$ outside the domain Ω, it will suffice to require that Eqs (1.9) hold in $R^2 \times \Omega$, rather than in all of R^5.

5. Equality (1.10) guarantees that the coefficient of ε^1 in the expression $(\nabla, \mathbf{v})$ in (1.1) will vanish.

6. To determine the asymptotic behaviour of the solution of problem (1.1), (1.2) to within $o(\varepsilon)$, it will suffice to consider as the flow $\mathbf{V}$ a vector field satisfying Eqs (1.1) to within the same accuracy. In particular, as the viscosity coefficient in (1.1) is $O(\varepsilon^2)$ and $\mathbf{V}$ is a smooth function of ε, the field $\mathbf{V}$ may be defined everywhere as a smooth solution of the Euler equations. Such flows will indeed be considered below as examples.

2. INVESTIGATION OF THE EQUATIONS OF A LAMINAR WAKE. THE OSEEN APPROXIMATION

We will now investigate the behaviour of the solutions of system (1.9) in the linear approximation. Dropping non-linear terms in (1.9) (i.e. assuming that the perturbation is small compared with ε), we obtain

$$\begin{aligned}&\dot{\mathbf{u}} + AT^{-1}\mathbf{u} + \mathbf{a}w + T\nabla_y\pi = \nu D^2\mathbf{u}, \quad (\nabla_y, \mathbf{u}) = 0\\&\dot{w} + (T^{-1}\mathbf{b}, \mathbf{u}) = \nu D^2 w\end{aligned} \tag{2.1}$$

The evolution with respect to t (i.e. along a streamline γ) of solutions of this system depends on the structure of the matrices A and T and the vectors $\mathbf{a}$ and $\mathbf{b}$, this structure may differ for different types of free-stream flow V.

In the simplest case—a constant flow $\mathbf{V} = (0, 0, V_0)$, V_0 = const—system (2.1) reduces to the parabolic equations

$$\dot{\mathbf{u}} = \nu\Delta_y\mathbf{u}, \quad \dot{w} = \nu\Delta_y w \quad (\nabla_y, \mathbf{u}) = 0 \tag{2.2}$$

whose solutions with initial data (1.12) have the following form (throughout, unless otherwise stated, integration will be performed over the whole space R^2)

$$\mathbf{u}(\mathbf{y},\alpha,t)=\frac{1}{4\pi\nu t}\int \mathbf{u}^0(\mathbf{y}',\alpha)\exp\left[\frac{-(\mathbf{y}-\mathbf{y}')^2}{4\nu t}\right]d^2\mathbf{y}'$$

$$w(\mathbf{y},\alpha,t)=\frac{1}{4\pi\nu t}\int w^0(\mathbf{y}',\alpha)\exp\left[\frac{-(\mathbf{y}-\mathbf{y}')^2}{4\nu t}\right]d^2\mathbf{y}'$$

Suppose that the surface Γ is the plane $x_3 = 0$, and that $S_j|_\Gamma = x_j$ ($j = 1, 2$). Then, obviously, $S_j(\mathbf{x}) = x_j$, $t(\mathbf{x}) = x_3/V_0$. Thus the perturbation (wake) has the form

$$\mathbf{U}=\mathbf{U}\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon},x_3\right)\equiv \mathbf{U}(\mathbf{y},x_3)=\frac{V_0}{4\pi\nu x_3}\int \mathbf{U}(\mathbf{y}',0)\exp\left[\frac{-(\mathbf{y}-\mathbf{y}')^2}{4\nu x_3}\right]d^2\mathbf{y}' \tag{2.3}$$

Equations (2.2) and formula (2.3) are well known in the theory of a laminar wake (the Oseen approximation). It follows from (2.3), in particular, that the amplitude of the perturbation $\mathbf{U}$ decreases as x_3 increases (owing to the viscosity).

The next two sections are devoted to investigating the evolution of the perturbation $\mathbf{U}$ in the linear approximation (i.e. the behaviour of the solutions of system (2.1) as $t \to \infty$) for more complicated flows, and to discussing the physical corollaries. An analogous analysis in the unsteady problem, describing the evolution of a narrow "peak" concentrated in the neighbourhood of a single point, has already been carried out [12–14].

For the rest of our calculations, it will be convenient to rewrite (2.1) as a system of equations in two unknown functions, w and $\psi(\mathbf{y}, \alpha, t)$, related to u by the formulae $u_1 = -i\partial\psi/\partial y_2$; $u_2 = i\partial\psi/\partial y_1$ ($i\psi$ is the stream function of the two-dimensional flow $\mathbf{u}$). Substituting the last formulae into (2.1), changing in these equations to Fourier transforms with respect to $\mathbf{y}$ and evaluating the scalar product of the first equation by the vector $T^{-1}\mathbf{n}'$ ($\mathbf{n} = (k_2, -k_1)$, where k_j are variables due to y_j and the prime denotes transposition), we obtain a system of ordinary differential equations

$$\begin{aligned}&\frac{d\tilde\psi}{dt}+\frac{(T^{-1}\mathbf{n},AT^{-1}\mathbf{n})}{(\mathbf{n},T^{-1}\mathbf{n})}\tilde\psi+\frac{(\mathbf{a},T^{-1}\mathbf{n})}{(\mathbf{n},T^{-1}\mathbf{n})}\tilde w=-\nu(\mathbf{k},T\mathbf{k})\tilde\psi\\&\frac{d\tilde\psi}{dt}+(\mathbf{b},T^{-1}\mathbf{n})\tilde\psi=-\nu(\mathbf{k},T\mathbf{k})\tilde w,\quad \mathbf{k}=(k_1,k_2)\end{aligned} \tag{2.4}$$

where $\tilde\psi(\mathbf{k})$ and $\tilde w(\mathbf{k})$ are the Fourier transforms of $\psi(\mathbf{y})$ and $w(\mathbf{y})$, respectively. Let $\psi(\mathbf{k}, \alpha, t)$, $\tilde w(\mathbf{k}, \alpha, t)$ be a solution of system (2.4) with initial data

$$\tilde w|_{t=0}=\tilde w_0(\mathbf{k},\alpha);\quad \tilde\psi|_{t=0}=\tilde\psi_0(\mathbf{k},\alpha)\quad (\tilde{\mathbf{u}}_0=\tilde\psi_0\mathbf{n}) \tag{2.5}$$

Then the solution of system (2.1) with initial data (1.12) will be

$$\begin{aligned}&\mathbf{u}(\mathbf{y},\alpha,t)=-i\left\|\begin{matrix}\partial/\partial y_2\\-\partial/\partial y_1\end{matrix}\right\|\frac{1}{2\pi}\int e^{i(\mathbf{k},\mathbf{y})}\tilde\psi(\mathbf{k},\alpha,t)d^2\mathbf{k}\\&w(\mathbf{y},\alpha,t)=\frac{1}{2\pi}\int e^{i(\mathbf{k},\mathbf{y})}\tilde w(\mathbf{k},\alpha,t)d^2\mathbf{k}\end{aligned} \tag{2.6}$$

Thus, to determine the evolution of the perturbation $\mathbf{U}$, one has to solve system (2.4) and then evaluate the integrals (2.6).

3. FLOWS WITH STRAIGHT STREAMLINES. INTERACTION OF LONGITUDINAL AND TRANSVERSE COMPONENTS OF THE VELOCITY IN THE WAKE

Unidirectional flows. Let the free-stream flow be $\mathbf{V}(\mathbf{x}) = (0, 0, V_0(x_1, x_2))$. The streamlines are straight lines parallel to the x_3 axis, the velocity of motion of the fluid particles varies from line to line. Let γ be the streamline $x_1 = x_2 = 0$, and let Γ be the plane $x_3 = 0$. Let $S_j^0 = x$, then, obviously, $S_j(\mathbf{x}) = x_j$. Elementary calculations give

$$\mathbf{u}(\mathbf{y},\mathbf{x})=\frac{V_0(\mathbf{x}_\perp)}{4\pi\nu x_3}\int \mathbf{u}_0(\mathbf{y}')\exp\left[-\frac{V_0}{4\nu x_3}(\mathbf{y}-\mathbf{y}')^2\right]d^2\mathbf{y}'$$

$$w(\mathbf{y},\mathbf{x})=\frac{V_0(\mathbf{x}_\perp)}{4\pi\nu x_3}\int w_0(\mathbf{y}')\exp\left[-\frac{V_0}{4\nu x_3}(\mathbf{y}-\mathbf{y}')^2\right]d^2\mathbf{y}'- \tag{3.1}$$

$$-\frac{1}{2}\int_0^{x_3} dz\int_{R^2} d^2\mathbf{y}'\frac{(\mathbf{u}(\mathbf{y}',\mathbf{x}_\perp,z),\nabla(V_0^2))}{4\pi\nu(x_3-z)}\exp\left[-\frac{V_0(\mathbf{y}-\mathbf{y}')^2}{4\nu(x_3-z)}\right]\quad \mathbf{x}_\perp=(x_1,x_2)$$

The last term describes the influence of the transverse components of the velocity in the wake ($\mathbf{u}$) on the longitudinal component (w). This influence—acceleration or deceleration of the jet—occurs because the velocity of motion of the fluid particles may differ on different streamlines of the free-stream flow $\mathbf{V}$. It is indeed obvious from (3.1) that transverse circulations in the direction of increasing velocity of the fluid particles $((\mathbf{u},\nabla)V_0^2>0)$ slow down the jet, while circulations in the direction of decreasing velocity $((\mathbf{u},\nabla)V_0^2<0)$ speed it up. Note that the whole perturbation is damped as $x_3\to\infty$ because of viscosity, while, since $(\nabla_y,\mathbf{u}_0)=0$, the transverse component u of the velocity decreases as $1/x_3^2$.

Plane-parallel flows with variable direction. Let the free-stream flow have the form $\mathbf{V}=(V_1(x_3), V_2(x_3),0)$. The streamlines are straight lines in the horizontal planes $x_3=\text{const}$; the flow direction changes from plane to plane. Let Γ be a plane passing through the x_3 axis and cutting the plane $x_3=0$ in a straight line orthogonal to the vector $\mathbf{V}(0)$. The equations of this plane are $x_3=\alpha_1$, $\mathbf{x}_{||}=\mathbf{W}(\alpha_1)\alpha_2$, where $\mathbf{x}_{||}=(x_1,x_2)$, $\mathbf{W}(x_3)=(V_2(x_3),-V_1(x_3))$. Let the streamline γ be the straight line $\mathbf{x}=\mathbf{V}(0)t$, and define the functions S_j^0 as $S_1^0=\mathbf{V}^2(\alpha_1)\alpha_2$; $S_2^0=\alpha$. Trajectories of the field V emanating from the plane Γ have the form $\mathbf{X}_{||}=\mathbf{V}_{||}(\alpha_1;t+\mathbf{W}(\alpha_1)\alpha_2, x_3=\alpha_1$. The calculations lead to the following solution of the first equation in (2.4)

$$\tilde{\psi}(k,\alpha,t)=\tilde{\psi}_0(k,\alpha)\frac{[k_1(\mathbf{V}_{||},\mathbf{W}')t+k_2+\alpha_2k_1(\mathbf{V},\mathbf{V}')]^2+\mathbf{V}^2k_1^2}{[k_2+\alpha_2k_1(\mathbf{V},\mathbf{V}')]^2+\mathbf{V}^2k_1^2}\times$$

$$\times\exp\left(-\nu\left\{\mathbf{V}^2k_1^2t+\frac{1}{3k_1(\mathbf{V}_{||},\mathbf{W}')}((k_1\alpha_2(\mathbf{V},\mathbf{V}')+k_2+k_1(\mathbf{V}_{||},\mathbf{W}')t)^3-(k_1\alpha_2(\mathbf{V},\mathbf{V}')+k_2)^3)\right\}\right)$$

Easy but cumbersome evaluations of the integrals (2.6) now show that the transverse components of the perturbations are

$$\mathbf{U}_\perp=\sum_{i,j=1}^{2}(T^{-1})_{ij}u_i(\nabla S_j)=O\frac{1}{\sqrt{t}}\quad \text{as}\quad t\to\infty$$

Thus, the perturbation decreases as $t\to\infty$ more slowly than for unidirectional flows, i.e. change of the flow direction obstructs viscous damping.

Finally, let us consider the interaction of the longitudinal and transverse components of the velocity in such flows. The equation for w is

$$\dot{w}-\nu D^2w=-(\mathbf{u},T^{-1}\mathbf{b})=-\frac{(\mathbf{V},\mathbf{V}')}{\mathbf{V}^2}u_2=-U_3\frac{\partial}{\partial x_3}\ln|\mathbf{V}|$$

Thus, if the transverse circulations in the jet are in the direction of increasing $|\mathbf{V}|$, the jet will slow down; otherwise the jet will accelerate, just as in the case of unidirectional free-stream flows.

4. SPIRAL FLOW. ENERGY JUMP IN THE WAKE. THE ANALOGUE OF RAYLEIGH'S INSTABILITY CONDITION

Let us consider the evolution of a laminar wake in a free-stream flow

$$\mathbf{V}=\omega(r)\left(\mathbf{e}_z+\frac{r}{r_0}\mathbf{e}_\varphi\right)=\left(-\frac{\omega}{r_0}x_2,\frac{\omega}{r_0}x_1,\omega\right),\quad r_0=\text{const}$$

where $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ are unit vectors of a cylindrical system of coordinates. The streamlines of this field are spiral curves

$$x_1 = r\cos\omega(t-t_0), \quad x_2 = r\sin\omega(t-t_0), \quad z = \omega t + z_0$$

(r = const on the trajectories). Equation (1.6) in cylindrical coordinates

$$\frac{\partial S_j}{\partial z} + \frac{1}{r_0}\frac{\partial S_j}{\partial \varphi} = 0$$

has the general solution $S_j = \Phi_j(r, \varphi - z/r_0)$, where $\Phi_j(r, \varphi)$ are arbitrary functions. Let us assume that these functions depend on only one variable, so that $S_1 = \Phi_1(r)$, $S_2 = \Phi_2(\varphi - z/r_0)$, where Φ_j are smooth functions, $\Phi_j' \neq 0$ in Ω. Calculating the coefficients in Eqs (2.4), we obtain

$$\frac{d}{dt}\tilde{\psi} - \frac{2k_2\omega^2 r\Phi_1'}{r_0^2 k^2\zeta^2}\tilde{w} = -\mathbf{k}^2\sigma\tilde{\psi}, \quad \frac{d}{dt}\tilde{w} + \frac{2k_2\xi\Phi_1'}{r_0^2}\tilde{\psi} = -\mathbf{k}^2\sigma\tilde{w} \tag{4.1}$$

where

$$\xi = r\omega^2 + \frac{1}{2}\omega\omega'(r_0^2 + r^2), \quad \mathbf{k}^2\zeta^2 = \frac{k_2^2}{\Phi_1'^2} + \frac{k_1^2}{\Phi_2'^2(r^{-2}+r_0^{-2})}$$

$$\sigma = \nu|\mathbf{k}|^{-2}(k_1^2(\Phi_1')^2 + k_2^2(\Phi_2')^2(r^{-2}+r_0^{-2}))$$

The solutions of this system depend on the time as $e^{(\pm i\lambda - \nu\mathbf{k}2\sigma)t}$ if

$$\xi = \frac{1}{4(r^2+r_0^2)}\frac{\partial}{\partial r}(\omega^2(r^2+r_0^2)^2) > 0 \quad \left(\lambda = \left|\frac{4k_2^2\omega^2 r(\Phi_1')^2\xi}{k^2\zeta^2 r_0^4}\right|^{1/2}\right)$$

and as $e^{(\pm\lambda - \nu\mathbf{k}2\sigma)t}$ if

$$\frac{\partial}{\partial r}(\omega^2(r^2+r_0^2)^2) < 0 \tag{4.2}$$

Assume that inequality (4.2) is true. We will investigate the behaviour as $t \to \infty$ of the perturbation $\mathbf{u}(\mathbf{y}, \alpha, t)$. Solving system (4.1) and omitting terms containing $e^{(-\lambda - \nu\mathbf{k}2\sigma)t}$, we obtain

$$\mathbf{u} = \frac{1}{4\pi}\int\begin{pmatrix}k_2\\-k_1\end{pmatrix}e^{\lambda t - \nu k^2\sigma t}\left(\tilde{\psi}_0(\mathbf{k}) + \frac{\beta\tilde{w}_0}{|\mathbf{k}|}(\mathbf{k})\right)e^{i(\mathbf{k},\mathbf{y})}d^2\mathbf{k}, \quad \beta = \sqrt{\frac{r\omega^2}{\zeta^2|\xi|}}$$

For simplicity, we will consider the behaviour of the perturbation on the wake axis, i.e. for $\mathbf{y} = 0$. Changing to polar coordinates ρ, θ in the last integral, substituting $\rho \to z = \rho\sqrt{(\nu\sigma t)}$, expanding $\tilde{\psi}_0(z/\sqrt{(\nu\sigma t)})$, $\tilde{w}_0(z/\sqrt{(\nu\sigma t)})$ by Taylor's formula and evaluating the integral with respect to dz in the leading terms, we obtain

$$\mathbf{u}(0) = \frac{1}{8\pi\nu t}\int_0^{2\pi}\begin{pmatrix}\sin\theta\\-\cos\theta\end{pmatrix}\frac{\beta(\theta)}{\sigma(\theta)}e^{t\lambda(\theta)}\Big(\tilde{w}_0(0) +$$

$$+\frac{\sqrt{\pi}}{2\sqrt{\nu\sigma t}}\left(\frac{\partial\tilde{w}_0}{\partial k_1}(0)\cos\theta + \frac{\partial\tilde{w}_0}{\partial k_2}(0)\sin\theta\right) + \frac{\sqrt{\pi}}{2\sqrt{\nu\sigma t}}\frac{\tilde{\psi}_0(0)}{\beta(\theta)} + O\left(\frac{1}{t}\right)\Big)\,d\theta$$

The function $\lambda(\theta)$ has a non-degenerate maximum at $\theta = \pi/2$ and $\theta = 3\pi/2$. Using Laplace's method, we obtain

$$\mathbf{u}(0) = \frac{C}{t^2}\frac{\partial\tilde{w}_0}{\partial k_2}(0)\begin{pmatrix}1\\0\end{pmatrix}e^{\lambda_0 t} + O(t^{-5/2}e^{\lambda_0 t}), \quad t \to \infty; \quad C = \frac{\sqrt{2}\beta_0}{8\nu^{3/2}\sigma_0^{3/2}\lambda_2^{1/2}} \tag{4.3}$$

where the subscript zero means that the function in question is evaluated at $\theta = \pi/2$.

It follows from (4.3) that, when condition (4.2) is satisfied, the perturbation U will not only not decay but will increase exponentially as $t \to \infty$, regardless of the viscosity (note that $|\nabla S_j|$ and the elements of the matrix Γ^{-1} are constant on trajectories of the free-stream flow). This effect—an energy jump in a jet—may be associated with the curvature of the streamlines of the free-stream flow; it is the curvature of the trajectories that enables the perturbation to "pick up" energy from the free-stream flow (as already remarked in Section 3; in flows with rectilinear trajectories the wake is damped). Note that when $r_0 = 0$ condition (4.2) is precisely Rayleigh's instability criterion (see, e.g. [13–16])—a condition that guarantees exponential growth of a three-dimensional short-wave perturbation in a two-dimensional radially symmetric flow $\mathbf{V} = r\omega\mathbf{e}_\varphi$. Of course, we are not concerned here with ordinary stability (in Lyapunov's sense or in the linear approximation), since the problem is a steady-state one. At the same time, the effect of the wake increasing along a trajectory is similar to instability; it may possibly point to the "irregular" organization of the set of stationary points in the phase space of the Navier–Stokes dynamical system.

Formula (4.3) also indicates a phenomenon in curved wakes: the longitudinal components of the velocity generate transverse components (only the reverse effect is observed in rectilinear flows; see Section 3). Under these conditions, the strongest growth in a spiral flow is that of the component u_1—the projection of the perturbation onto the radial direction. Thus, as the distance from the body in the flow increases, the wake is "pulled out" along the axis of the cylinder on which the spiral trajectory of the free-stream flow lies.

REFERENCES

1. MASLOV, V. P., Coherent structures, resonances and asymptotic non-uniqueness for the Navier–Stokes equations at high Reynolds numbers. *Uspekhi Mat Nauk*, 1986, **41**, 19–35.
2. MASLOV, V. P., *Asymptotic Methods for Solving Pseudo-differential Equations*. Nauka, Moscow, 1987.
3. BLOEMBERGEN, N., *Nonlinear Optics*. Benjamin, New York, 1965.
4. AKHMANOV, S. A. and KHOKHLOV, R. V., *Problems of Non-linear Optics Veseoyuz.* Inst. Nauch. Tekhn. Inform., Moscow, 1964.
5. MASLOV, V. P. and OMEL'YANOV, G. A., Interaction of three waves taking frequency-doubling effects into account. *Izv. Vuzov. Fizika*, 1986, **29**, 3–23.
6. MASLOV, V. P. and OMEL'YANOV, G. A., Equations of the Kadomtsev–Pogutse type for a tokamak and for domains with arbitrary symmetry. *Dokl. Ross. Akad. Nauk*, 1992, **326**, 83–90.
7. MASLOV, V. P. and OMEL'YANOV, G. A., Rapidly oscillating asymptotic solutions of the equations of magnetohydrodynamics in the tokamak approximation. *Teoret. Mat. Fiz.*, 1992, **92**, 879–895.
8. MASLOV, V. P. and OMEL'YANOV, G. A., Three-scale expansion of solutions of the equations of magnetohydrodynamics and Reynolds equation for a tokamak. *Teoret. Mat. Fiz.*, 1994, **98**, 297–311.
9. MASLOV, V. P. and OMEL'YANOV, G. A., Fluctuation-generated tokamak pinch instabilities. *Russ. J. Math. Phys.*, 1994, **2**, 463–485.
10. KADOMTSEV, B. B. and POGUTSE, O. P., Non-linear spiral perturbations of plasma in a tokamak. *Zh. Eksp. Teoret. Fiz.*, 1973, **65**, 575–589.
11. PUSTOVITOV, V. D. and SHAFRANOV, V. D., Plasma equilibrium and plasma stability in stellarators. *Voprosy Teor. Plazmy (Energoatomizdat)*, 1987, **15**, 146–291.
12. DOBROKHOTOV, S. Yu. and SHAFAREVICH, A. I., Parametrics and asymptotic forms of localized solutions of the Navier–Stokes equations in R^3, linearized on a smooth flow. *Mat. Zametki*, 1992, **51**, 72–82.
13. DOBROKHOTOV, S. Yu. and SHAFAREVICH, A. I., Some asymptotic solutions of the linearized Navier–Stokes equations. *Mat. Zametki*, 1993, **53**, 25–35.
14. SHAFAREVICH, A. I., The behaviour as $t \to \infty$ of rapidly decreasing asymptotic solutions of the linearized Navier–Stokes equations. *Mat. Zametki*, 1994, **55**, 124–145.
15. RAYLEIGH, J. W. S., On the dynamics of revolving fluids. *Proc. Roy. Soc. London Ser. A*, 1917, **93**, 148–154.
16. BAYLY, B. J., Three-dimensional centrifugal-type instabilities in inviscid two-dimensional flows. *Physics of Fluids*, 1988, **31**, 56–64.

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